Miscellaneous Exercise 11

2a. $f(x) = 4 - x^2$ $(0, 4) \cap$

D : $x \in R$

R : $f(x) \leq 4$

b. $f(x) = (x+3)^2 - 7$ $(-3, -7) \cup$

D : $x \in R$

R : $f(x) \geq -7$

c. $f(x) = \sqrt{x+2}$

D : $x + 2 \geq 0$

$x \geq -2$, $x \in R$

R : $f(x) \geq 0$

d. $f(x) = 5x + 6$

D : $x \in R$

R : $f(x) \in R$

e. $f(x) = (2x+3)^2$ $\vee \left(-\frac{3}{2}, 0\right)$

D : $x \in R$

R : $f(x) \geq 0$

f. $f(x) = 2 - \sqrt{x}$

D : $x \geq 0$

R : $f(x) \leq 2$

5. $f(x) = 2x^2 - 8x + 10$, $0 \leq x \leq 2$

(i) $= 2(x^2 - 4x) + 10$

$= 2[(x-2)^2 - 4] + 10$

$= 2(x-2)^2 - 8 + 10$

$= 2(x-2)^2 + 2$

Vertex $(2, 2)$

(ii) $f(0) = 10$, $f(2) = 2$ } Range : $2 \leq f(x) \leq 10$

(iii) Domain of $f^{-1}(x)$

$2 \leq x \leq 10$ [R of ori]

(iv) $y = 2(x-2)^2 + 2$

$y - 2 = 2(x-2)^2$

$\frac{y-2}{2} = (x-2)^2$

$x - 2 = \pm\sqrt{\frac{y-2}{2}}$

$x = 2 \pm \sqrt{\frac{y-2}{2}}$

$f^{-1}(x) = 2 - \sqrt{\frac{x-2}{2}}$ //

7 $f(x) = 4x - 2k$

$g(x) = \frac{9}{2-x}$

i $fg(x) = f\left(\frac{9}{2-x}\right)$

$= 4\left(\frac{9}{2-x}\right) - 2k = x$

$\frac{36}{2-x} = 2k + x$

$36 = (2-x)(2k+x)$

$-x^2 + 2x - 2kx + 4k - 36 = 0$

$-x^2 + (2-2k)x + 4k - 36 = 0$

$D = 0$ (2 equal roots)

$(2-2k)^2 - 4(-1)(4k-36) = 0$

$4 - 8k + 4k^2 + 16k - 144 = 0$

$4k^2 + 8k - 140 = 0$

$k^2 + 2k - 35 = 0$

$(k+7)(k-5) = 0$

$k = -7, k = 5$

ii $k = -7$: $-x^2 + (2+14)x - 28 - 36 = 0$

$-x^2 + 16x - 64 = 0$

$x^2 - 16x + 64 = 0$

$(x-8)^2 = 0$, $x = \underline{\underline{8}}$

$k = 5$: $-x^2 + (2-10)x + 20 - 36 = 0$

$-x^2 - 8x - 16 = 0$

$x^2 + 8x + 16 = 0$

$(x+4)^2 = 0$, $x = \underline{\underline{-4}}$

8 $f(x) = 2x + 1$, $x > 0$

$g(x) = \frac{2x-1}{x+3}$, $x \neq -3$

i $gf(x) = x$

$g(2x+1) = x$

$\frac{2(2x+1)-1}{2x+1+3} = x$

$\frac{4x+1}{2x+4} = x$

$4x + 1 = 2x^2 + 4x$

$2x^2 - 1 = 0$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}} \Rightarrow x = \underline{\underline{\sqrt{\frac{1}{2}}}}$

ii $f^{-1}(x) = \frac{x-1}{2}$

$g(x) = \frac{2x-1}{x+3}$

$yx + 3y = 2x - 1$

$(y-2)x = -3y - 1$

$x = \frac{-3y-1}{y-2} = \frac{1+3y}{2-y}$

$g^{-1}(x) = \frac{1+3x}{2-x}$ //

iii $g^{-1}(x) = x$

$\frac{1+3x}{2-x} = x$

$2x - x^2 = 1 + 3x$

$x^2 + x + 1 = 0$

$D = 1 - 4\,1\,1 = -3$

$D < 0 \Rightarrow$ No Solutions

9 $f(x) = 2x + 1$

$g(x) = x^5$

$h(x) = \frac{1}{x}$

a $x \to (2x+1)^5$

gf

b $x \to 4x + 3$

$2(2x+1) + 1$

ff

c $g^{-1}(x)$

d $x \to 2x^{-5} + 1$

fhg

e $x \to \frac{1}{2x^5 + 1}$

$fg = 2x^5 + 1$

hfg

f. $x \to \frac{x-1}{2}$

$f^{-1}(x)$

g $x \to \sqrt[5]{\frac{2}{x^5} + 1}$

$g^{-1}fhg$

h $x \to \frac{2}{x-1}$

hf^{-1}

10 $f(x) = \begin{cases} 3x-2 & \text{for } -1 \leq x \leq 1 \\ \frac{4}{5-x} & \text{for } 1 < x \leq 4 \end{cases}$

i Range: $x = -1, f(x) = -5$

$x = 4, f(x) = 4$

$-5 \leq f(x) \leq 4$

iii $y = \frac{4}{5-x}$

$\frac{1}{y} = \frac{5-x}{4}$

$5 - x = \frac{4}{y}$

$5 - \frac{4}{y} = x$

$f^{-1}(x) = \begin{cases} \frac{x+2}{3}, & -5 \leq x \leq 1 \\ \frac{5x-4}{x}, & 1 < x \leq 4 \end{cases}$

11 $f(x) = x^2 + 6x$

$g(x) = 2x - 1$

$fg(x) = f(2x-1) = (2x-1)^2 + 6(2x-1)$

$= 4x^2 - 4x + 1 + 12x - 6$

$= 4x^2 + 8x - 5$

$gf(x) = g(x^2+6x)$

$= 2(x^2+6x) - 1$

$= 2x^2 + 12x - 1$

$4x^2 + 8x - 5 = 2x^2 + 12x - 1$

$2x^2 - 4x - 4 = 0$

$x^2 - 2x - 2 = 0$

$(x-1)^2 - 1 - 2 = 0$

$x - 1 = \pm\sqrt{3}$

$x = 1 \pm \sqrt{3}$

12 $f(x) = x^2 - 2x + 7 \quad x \leq k$

$= (x-1)^2 - 1 + 7 \quad k = 1$

$= (x-1)^2 + 6$

$f(x) \geq 6$

$y = (x-1)^2 + 6$

$y - 6 = (x-1)^2$

$\pm\sqrt{y-6} = x - 1$

$x = 1 \pm \sqrt{y-6}$

$f^{-1}(x) = 1 - \sqrt{x-6}$

Domain: $x \geq 6$

Range: $f^{-1}(x) \leq 1$

13 $f(x) = x^2 + 2x + 3$

$g(x) = ax + b$

$fg(x) = 4x^2 - 48x + 146$

$f(ax+b) = 4x^2 - 48x + 146$

$(ax+b)^2 + 2(ax+b) + 3 = 4x^2 - 48x + 146$

$a^2x^2 + 2abx + b^2 + 2ax + 2b + 3 = 4x^2 - 48x + 146$

$a^2x^2 + (2ab + 2a)x + b^2 + 2b + 3 = 4x^2 - 48x + 146$

$b^2 + 2b + 3 = 146$

$a^2 = 4$

$b^2 + 2b - 143 = 0$

$a = \pm 2$

$(b-11)(b+13) = 0$

$a = 2$:

$2a(b+1) = -48$

$4(b+1) = -48$

$b + 1 = -12$

$b = -13$

$a = -2$:

$2a(b+1) = -48$

$-4(b+1) = -48$

$b + 1 = 12$

$b = 11$

14 $f(x) = 2x+1$

$g(x) = x^2-2$

i $fg(x) = f(x^2-2)$

$= 2(x^2-2)+1$

$= 2x^2-3$

$gf(x) = g(2x+1)$

$= (2x+1)^2-2$

$= 4x^2+4x-1$

ii $fg(a) = gf(a)$

$2a^2-3 = 4a^2+4a-1$

$2a^2+4a+2 = 0$

$a^2+2a+1 = 0$

$(a+1)^2 = 0$

$a = -1$

iii $g(b) = b^2-2 = b$

$b^2-b-2 = 0$

$(b-2)(b+1) = 0$

$b = 2,\ b = -1$

iv $f^{-1}g(x)$

$f^{-1} = \frac{x-1}{2}$

$f^{-1}(x^2-2) = \frac{x^2-2-1}{2}$

$= \frac{x^2-3}{2}$

v $h(x) = x^2-2,\ x \le 0$

$h^{-1}(x) = -\sqrt{x+2}$

15 $f(x) = 4x+5$, $g(x) = 3-2x$

a $f^{-1} = \frac{x-5}{4}$

b $y = 3-2x$

$2x = 3-y$

$x = \frac{3-y}{2}$

$g^{-1}(x) = \frac{3-x}{2}$

c. $f^{-1}g^{-1} = f^{-1}\left(\frac{3-x}{2}\right)$

$= \frac{\frac{3-x}{2}-5}{4}$

$= \frac{3-x-10}{8} = \frac{-x-7}{8}$

d $gf = g(4x+5)$

$= 3-2(4x+5)$

$= 3-8x-10$

$= -7-8x$

e. $(gf)^{-1} =$

$y = -7-8x$

$8x = -7-y$

$x = \frac{-7-y}{8}$

$(gf)^{-1} = \frac{-7-x}{8} = f^{-1}g^{-1}$

17 $f(x) = 10-x$

a $f(7) = 10-7 = 3$

b $ff(7) = f(3) = 7$

c $f^{15}(7) = 3$

d $f^{100}(7) = 7$

18 $f(x) = \frac{x+5}{2x-1}$

a $ff = f\left(\frac{x+5}{2x-1}\right) = \frac{\frac{x+5}{2x-1}+5}{2\left(\frac{x+5}{2x-1}\right)-1} = \frac{(x+5+10x-5)}{2x-1} \times \frac{2x-1}{2x+10-2x+1}$

$= \frac{11x}{11} = x$

b $f^3 = f(x) = \frac{x+5}{2x-1}$

c $f^4 = x$

d. $f^{10} = x$

e $f^{351} = \frac{x+5}{2x-1}$

19. $f^{-1}(x) = \frac{2x-4}{x}$

$yx = 2x-4$

$4 = 2x-yx$

$4 = x(2-y)$

$x = \frac{4}{2-y}$

a $f(x) = \frac{4}{2-x}$

a. $ff = f\left(\frac{4}{2-x}\right) = \frac{4}{2-\frac{4}{2-x}} = \frac{4(2-x)}{4-2x-4} = \frac{4(x-2)}{2x} = \frac{2(x-2)}{x}$

c $f^3 = f\left(\frac{2x-4}{x}\right) = \frac{4}{2-\frac{2x-4}{x}} = \frac{4x}{4} = x$

d $f^4 = f(x) = \frac{4}{2-x}$

$f = f^4$

$ff = \frac{2x-4}{x}$

$f^3 = x$

e $f^{12} = f^3 = x$

f $f^{82} = f = \frac{2x-4}{x} = \frac{4}{2-x}$

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